

REFLECTIONLESS MEASURES WITH SINGULAR COMPONENTS

F. NAZAROV, A. VOLBERG, P. YUDITSKII

1. INTRODUCTION AND MAIN RESULTS

The Cauchy transform (potential) $\mathcal{C}_\nu(z)$ of a Radon measure ν in \mathbb{C} is defined by

$$\mathcal{C}_\nu(z) := \int \frac{d\nu(\zeta)}{\zeta - z}.$$

We consider here only **real** ν supported on compact $E \subset \mathbb{R}$, and call this measure reflectionless if there exists a limit $\mathcal{C}_\nu(x + i0)$ and it is purely imaginary for ν almost every $x \in E$. We call this measure weakly reflectionless if there exists a limit $\mathcal{C}_\nu(x + i0)$ and it is purely imaginary for dx almost every $x \in E$.

We need two classes of compact sets. First is the homogeneous sets: $|E \cap (x-h, x+h)| \geq \eta h$ for all $x \in E$ and all $h \in (0, \text{diam} E]$. Such sets have many interesting properties. Among those are:

- If ω denotes the harmonic measure of $\Omega := \mathbb{C} \setminus E$ then ω is mutually absolutely continuous with $dx|_E$ and $d\omega/dx \in L^p$, $p > 1$. The exponent p depends on η above.
- If ℓ is any bounded complementary interval of compact E (there is also one unbounded interval “centered” at infinity), and c_ℓ is a maximum point for Green’s function $G(z) := G(z, \infty)$ of Ω then

$$\sum_{\ell} G(c_\ell) < \infty. \quad (1.1)$$

- If $\nu = wdx + \nu_s$ is a real measure, ν_s its singular part, and $\mathcal{C}_\nu(x + i0)$ exists and it is purely imaginary for $dx|_E$ almost every $x \in E$ (so ν is weakly reflectionless) then $\nu_s = 0$.

These facts can be found in [SoYu], [Z], directly or under a small disguise.

The sets which satisfy the property (1.1) alone are called Widom sets (and Ω is called Widom domain).

Our goal is to show that there exists a Widom set E and a positive measure $\nu = wdx + \delta_{b_0}$ such that ν is weakly reflectionless. The obvious candidate for weakly reflectionless measure is the harmonic measure. In fact, here is a theorem which is proved (practically) in [MPV]:

Theorem 1.1. *Let E, ω be a compact set on \mathbb{R} and harmonic measure of $\mathbb{C} \setminus E$ with pole at infinity. We do not assume homogeneity or any other property on E . Then $\mathcal{C}_\omega(x + i0)$ exists and it is purely imaginary for $dx|_E$ almost every $x \in E$. So harmonic measure ω is **always** weakly reflectionless.*

Notice that theorem is not empty only if $|E| > 0$.

But for Widom domains we cannot hope to have harmonic measure as having both a) weak reflectionless property and b) non-trivial singular part. This is because of the

Theorem 1.2. *For Widom E , harmonic measure is always mutually absolutely continuous with respect to $dx|E$.*

(Of course harmonic measure of anything cannot have a point mass, anyway.)

2. EXAMPLE.

Example. We start with segment $[b_0, a_0]$ on the real axis, and we throw away open intervals $l_j := (a_j, b_j)$ such that l_m is to the left from l_{m-1} and they accumulate to b_0 from the right. We call l_j gaps, and we call $[b_{j+1}, a_j]$ slits. $E := \cup_j [b_{j+1}, a_j] \cup \{b_0\}$. In the future j -th slit will be very small with respect to j -th gap. Here is a function analytic in $\mathbb{C} \setminus E$:

$$R(z) = -\frac{1}{\sqrt{(z-a_0)(z-b_0)}} \prod_{j=1}^{\infty} \frac{z-x_j}{\sqrt{(z-a_j)(z-b_j)}}.$$

Properties of R : 1) R is purely imaginary on slits $[b_{j+1}, a_j]$; 2) R is real on gaps (a_j, b_j) , and it is positive on (a_j, x_j) and negative on (x_j, b_j) ; 3) $\Im R(z) \geq 0$ if $\Im z \geq 0$ (Nevanlinna function).

Therefore,

$$R(z) = \int \frac{d\mu(x)}{x-z},$$

where μ is a positive measure on E .

Notice the following

$$\Im[(z-b_0)R(z)] \geq 0 \text{ if } \Im z \geq 0. \quad (2.1)$$

In fact,

$$(z-b_0) \int_E \frac{d\mu(x)}{x-z} = \int_E \frac{(z-x)d\mu(x)}{x-z} + \int_E \frac{(x-b_0)d\mu(x)}{x-z} = -\|\mu\| + \int_E \frac{d\nu(x)}{x-z},$$

where $\nu = (x-b_0)\mu$ is a positive measure as $x-b_0 \geq 0$ on E . This proves (2.1).

But then $(z-b_0)R(z)$ also has the following property:

$$(x-b_0)R(x) < 0, \text{ for } x < b_0, \text{ or } x > a_0. \quad (2.2)$$

Therefore, $\log[-(x-b_0)R(x)]$ has compactly supported imaginary part called $-g(x)$. In fact, by (2.2) $g(x) = 0$ on $(-\infty, b_0) \cup (a_0, \infty)$. Then in the upper half-plane \mathbb{C}_+ we have

$$\log[-(z-b_0)R(z)] = -\frac{1}{\pi} \int \frac{g(x) dx}{x-z}.$$

and hence,

$$(z-b_0)R(z) = -e^{-\frac{1}{\pi} \int \frac{g(x) dx}{x-z}}. \quad (2.3)$$

Notice that all these minuses are really essential. By the definition of g we know not only that it has a compact support but also we know that $g = \pi/2$, where R is purely imaginary and that $g = \pi$, when R is negative. Otherwise g is zero. So, $g \neq 0$ on $E \cup \cup_{j=1}^{\infty} (x_j, b_j) = \cup_{j=1}^{\infty} (x_{j+1}, a_j)$.

Now μ has a point mass at b_0 iff

$$\lim_{x \rightarrow b_0^-} (x-b_0)R(x) > 0.$$

We use (2.3) to conclude that this may happen iff

$$\int \frac{g(x) dx}{x - z} < +\infty.$$

But we just calculated g , and clearly the last condition is equivalent to

$$\sum_j \frac{a_j - x_{j+1}}{x_{j+1} - b_0} < \infty. \quad (2.4)$$

We can easily take $x_j = b_j$ for all j , and transform that to

$$\sum_j \frac{a_j - b_{j+1}}{b_{j+1} - b_0} < \infty. \quad (2.5)$$

Now we will show that condition (2.5) is compatible with the Widom property of $\Omega = \mathbb{C} \setminus E = \mathbb{C} \setminus (\cup_{j=1}^{\infty} [b_{j+1}, a_j] \cup \{b_0\})$.

Given a domain Ω , its harmonic measure $\omega(E, z)$ means harmonic measure of $E \subset \partial\Omega$ at pole $z \in \Omega$.

Choose

$$b_0 = 0, a_0 = 1.$$

We will choose $a_j, b_j, j \geq 1$ inductively. Let $\ell_1 = (a_1, b_1), \dots, \ell_{n-1} = (a_{n-1}, b_{n-1})$ be chosen. Consider $\Omega_{n-1} := \mathbb{C} \setminus ([b_1, a_0] \cup [b_2, a_1] \cup \dots \cup [b_{n-1}, a_{n-2}])$. We consider also $\Omega_n = \Omega_{n-1} \setminus [b_n, a_{n-1}]$, where b_n is chosen as follows: we denote corresponding harmonic measures by ω_{n-1}, ω_n and require that

$$\omega_n([b_n, a_{n-1}], 0) \geq \frac{1}{2}. \quad (2.6)$$

Now we choose $a_n > 0$ and very close to 0. Then we choose b_{n+1} by the criterion (2.6), namely, to have

$$\omega_{n+1}([b_{n+1}, a_n], 0) \geq \frac{1}{2}. \quad (2.7)$$

Obviously we need

Lemma 2.1. *Let E be a compact subset of the positive half-axis. Let $I = [b', a]$ be a segment between 0 and E , such that for $\Omega = \mathbb{C} \setminus (E \cup I)$ we have $\omega(I, 0) \geq 12$. Then*

$$\lim_{a \rightarrow 0} \frac{b'}{a} = 1.$$

Suppose the lemma is proved. Then we can finish our construction. In fact, Choose a_n from (2.7) so close to 0 that for b_{n+1} from (2.7) one has

$$\frac{b_{n+1}}{a_n} \in (1 - 2^{-n}, 1).$$

Then condition (2.5): $\sum_j \frac{a_j - b_{j+1}}{b_{j+1} - b_0} < \infty$ is automatically satisfied (recall that $b_0 = 0$ in our construction).

We are left to check that (2.6) for all n ensures that the domain $\Omega = \lim \Omega_n = \mathbb{C} \setminus ([b_1, a_0] \cup [b_2, a_1] \cup \dots \cup [b_{n-1}, a_{n-2}] \cup \dots)$ is a Widom domain.

Let G_n, G are Green's function of Ω_n, Ω with pole at infinity. Let c_k be the critical points of G , c_k is its unique maximum in $\ell_k = (a_k, b_k)$. So $c_n > 0$. Obviously

$$G_n(c_n) \geq G(c_n)$$

by the principle of the extension of the domain. But clearly $G_n(x)$ is a decreasing function on $(-\infty, b_n]$. Therefore,

$$G_n(0) \geq G_n(c_n)$$

Two last inequalities together show that series $\sum_n G(c_n)$ converges (Widom's condition) if series $\sum_n G_n(0)$ converges. So we are left to see why the fact that (2.6) holds for all n implies that series $\sum_n G_n(0)$ converges. We write the Poisson integral for $x \in \Omega_n$:

$$G_{n-1}(x) - G_n(x) = \int_{[b_n, a_{n-1}]} G_{n-1}(y) d\omega_n(y, x).$$

Now we use Harnack's inequality to conclude

$$\tau G_{n-1}(0) \leq G_{n-1}(y) \quad \forall y \in [b_n, a_{n-1}].$$

Here τ is an absolute constant. In fact, all points of the segment $[b_n, a_{n-1}]$ are much closer to 0 than to $\partial\Omega_{n-1} = [b_{n-1}, a_{n-2}] \cup [b_{n-2}, a_{n-3}] \cup \dots \cup [b_1, a_0]$. Therefore,

$$G_{n-1}(0) - G_n(0) \geq \tau \int_{[b_n, a_{n-1}]} G_{n-1}(0) d\omega_n(y, x) = \tau \omega_n([b_n, a_{n-1}], 0) G_{n-1}(0) \geq \frac{\tau}{2} G_{n-1}(0).$$

We get

$$G_n(0) \leq (1 - \frac{\tau}{2}) G_{n-1}(0) \leq \dots \leq C (1 - \frac{\tau}{2})^{n-1}. \quad (2.8)$$

We proved the convergence of $\sum G(c_n) \leq \sum G_n(0)$: the Widom property.

Now we prove Lemma 2.1.

Proof. Without loss of generality we can think that $E \subset [1, \infty)$, $0 < b' < a < 1$. We also may consider special $\Omega = \mathbb{C} \setminus ([b', a] \cup [1, \infty))$, the general case follows immediately by the principle of extension of the domain. Now let $k = b'/a$. Rescale the domain to get $\Omega_k = \mathbb{C} \setminus ([k, 1] \cup [1/a, \infty))$. Keep k fixed and let $a \rightarrow 0$. Then obviously $\omega_{\Omega_k}([1/a, \infty), 0) \rightarrow 0$. So for some $a(k)$ $\omega_{\Omega_k}([k, 1], 0) \geq \frac{1}{2}$. Rescaling back, we see that we can choose k as close to 1 as we wish, and choose $a(k)$ in such a way that

$$\omega_{\Omega}([b', a], 0) \geq \frac{1}{2}.$$

We are done. □

3. HARMONIC MEASURES ARE WEAKLY REFLECTIONLESS.

In [MPV] the following result is proved: Let E be a compact subset of the real line. Let harmonic measure ω of $\mathbb{C} \setminus E$ be absolutely continuous with respect to dx on E . Then ω is such that $C_{\omega}(x + i0)$ is 0 dx (and hence $d\omega$) almost everywhere on E .

We show here by a simple way that a more general result holds:

Theorem 3.1. *Let E be a compact subset of the real line. Then ω is such that $C_{\omega}(x + i0)$ is 0 dx almost everywhere on E .*

Proof. Let ℓ_n be complementary intervals to E with ℓ_0 being the interval containing infinity. Consider $\Omega_n := (\mathbb{C} \setminus \mathbb{R}) \cup \ell_1 \cup \dots \cup \ell_n$. Let b_0 be the leftmost point of E , a_0 being the rightmost point of E . We use the notations from above. Let

$$R_n(z) = \int \frac{d\omega_n(t)}{t - z},$$

we know that

$$(x - b_0)R_n(x) < 0, \text{ for } x < b_0, \text{ or } x > a_0. \quad (3.1)$$

And $(z - b_0)R_n(z)$ is a Nevanlinna function (positive imaginary part in \mathbb{C}_+). Therefore, we can define logarithm of $-(z - b_0)R_n(z)$ in the upper half plane, and the imaginary part of this logarithm has compact support on the real line. Call it $-g_n(x)$. Then (for $z \in \mathbb{C}_+$)

$$R_n(z) = -\frac{1}{z - b_0} e^{-\int \frac{g_n(t)dt}{t-z}}.$$

Similarly,

$$R(z) = -\frac{1}{z - b_0} e^{-\int \frac{g(t)dt}{t-z}}. \quad (3.2)$$

Of course compactly supported (on $[b_0, a_0]$) functions g_n have values in $[0, \pi]$, and $g_n \rightarrow g$ weakly in L^2 . In fact, $R_n(z)$ converges to $R(z)$ uniformly on compact sets in $\mathbb{C} \setminus \mathbb{R}$ (this is just weak convergence of ω_n to ω). Then $\int \frac{g_n(t)dt}{t-z} \rightarrow \int \frac{g(t)dt}{t-z}$ uniformly on compact sets. At the same time g_n and g are compactly supported and have L^∞ norm bounded by π . Therefore, indeed $g_n \rightarrow g$ weakly in L^2 . But then certain convex combinations of g_n converge to g a.e. with respect to dx . All these convex combinations are identically equal to $\frac{\pi}{2}$ on E . We conclude that $g = \pi/2$ a.e. on E with respect to Lebesgue measure. Now formula (3.2) shows that $C_\omega(x + i0) = 0$ Lebesgue a.e. on E . We are done. \square

4. SINGULAR CONTINUOUS COMPONENTS OF WEAKLY REFLECTIONLESS MEASURES.

Now we are going to construct the closed set $E = [b_0, a_0] \setminus \cup(a_j, b_j)$ such that function

$$R(z) = -\frac{1}{\sqrt{(z - a_0)(z - b_0)}} \prod_{j=1}^{\infty} \frac{z - x_j}{\sqrt{(z - a_j)(z - b_j)}} =: \int_E \frac{d\mu(t)}{t - z}$$

has a singular continuous component in measure μ . We construct E inductively. Let $b_0 = -1, a_0 = 1$. Let us choose very fast decreasing to zero sequence ε_n , and let sequence L_n be such that $L_{n+1}/L_n < \varepsilon_n$. The set E will be comprised of closed segments and the limit points.

Step 1.

First two segments $s_1 := [b_0, c_0], s_2 := [d_0, a_0]$ are such that for $z \in n_1 := [-L_1/2, L_1/2]$ we have

$$1 - \varepsilon_1 < \left| \frac{z - d_0}{z - a_0} \right|^{\frac{1}{2}} \left| \frac{z - c_0}{z - b_0} \right|^{\frac{1}{2}} < 1.$$

Given a segment $s = [\alpha, \beta]$ we put $r_s(z) := \left| \frac{z - \alpha}{z - \beta} \right|^{\frac{1}{2}}$. We put $p_1 = c_0, p_2 = d_0$. We choose now two symmetric segments s_3, s_4 such that $s_3 < 0, s_4 > 0$ and

- $r_{s_i}(0) \in (1 - \varepsilon_2, 1 + \varepsilon_2), i = 3, 4$;
- Green's function G_4 of $\Omega_4 := \mathbb{C} \setminus \cup_{i=1}^4 s_i$ is smaller than ε_2 between s_3, s_4 .

We know by Section 2 that we just need to have segments s_3, s_4 close enough to zero.

Now we choose interval $n_2 = (ln_2, rn_2)$ centered at zero of length $\leq L_2$ and such that

$$r_{s_i}(z) \in (1 - \varepsilon_2, 1 + \varepsilon_2), i = 3, 4, \forall z \in n_2.$$

We put p_3 at the right endpoint of s_3 and p_4 at the left endpoint of s_4 .

To finish step 1 we choose segment s_5 centered at zero, inside n_2 and of such a small length $2\ell_1$ that

$$r_{s_5}(z) \in (1 - \varepsilon_3, 1 + \varepsilon_3) \quad z = rn_2/2, ln_2/2.$$

Put $E_5 := \cup_{i=1}^5 s_i$. Notice that

- Green's function G_5 of $\Omega_5 := \mathbb{C} \setminus E_5$ is smaller than ε_2 between s_3, s_4 .
- Measure μ_5 built by our formula on E_5 by assignment of four points p_1, \dots, p_4 has very large mass on s_5 , in fact

$$\mu_5(s_5) \geq (1 - \varepsilon_1)(1 - \varepsilon_2)^2 \int_{-\ell_1}^{\ell_1} \frac{dx}{\sqrt{\ell_1^2 - x^2}}. \quad (4.1)$$

Step 2.

Now we choose interval $n_3 = (ln_3, rn_3), n_4 = (ln_4, rn_4)$ centered at $ln_2/2$ and $rn_2/2$ of length $\leq L_3$ and such that

$$r_{s_5}(z) \in (1 - \varepsilon_3, 1 + \varepsilon_3), \quad \forall z \in n_3 \cup n_4 \cup (\mathbb{R} \setminus (ln_2/4, rn_2/4)). \quad (4.2)$$

This is only one requirement on n_3, n_4 . We will list another. Put p_5, p_6 as left and right endpoints of s_5 . Choose pair of segments s_6, s_7 symmetrically around ln_2 , and s_8, s_9 symmetrically around rn_2 . We do this so that

- $r_{s_i}(ln_2) \in (1 - \varepsilon_3, 1 + \varepsilon_3), i = 6, 7$;
- $r_{s_i}(rn_2) \in (1 - \varepsilon_3, 1 + \varepsilon_3), i = 8, 9$;
- $s_6, s_7, s_8, s_9 \subset n_2$;
- $r_{s_i}(z) \in (1 - \varepsilon_3, 1 + \varepsilon_3), i = 6, 7, 8, 9, \text{ for all } z \in \mathbb{R} \setminus n_2$;
- Green's function G_9 of $\Omega_9 := \mathbb{C} \setminus \cup_{i=1}^9 s_i$ is smaller than ε_3 between s_6, s_7 and between s_8, s_9 .

We know by Section 2 that we can reconcile the first four items with the fifth one: we just need to have segments s_6, s_7 with small $r_{s_i}(ln_2)$, $i = 6, 7$, close enough to ln_2 and s_8, s_9 with small $r_{s_i}(rn_2)$, $i = 8, 9$, close enough to rn_2 .

Now we have the second requirement on lengths of n_3, n_4 :

- $r_{s_i}(z) \in (1 - \varepsilon_3, 1 + \varepsilon_3), i = 6, 7, \quad \forall z \in n_3 \cup (\mathbb{R} \setminus n_2)$;
- $r_{s_i}(z) \in (1 - \varepsilon_3, 1 + \varepsilon_3), i = 8, 9, \quad \forall z \in n_4 \cup (\mathbb{R} \setminus n_2)$;

To finish step 2 we denote by ln_{51}, rn_{51} points which are mid-points of the left and the right halves of segment n_3 correspondingly. By ln_{52}, rn_{52} points which are mid-points of the left and the right halves of segment n_4 correspondingly.

Given interval J and number $\lambda > 0$, we denote (as usual) by λJ the interval with the same center and the length $\lambda|J|$.

We choose segments s_{51}, s_{52} centered at $ln_2/2, rn_2/2$ correspondingly, inside n_3, n_4 correspondingly, and of such a small length $2\ell_2$ that

$$r_{s_{51}}(z) \in (1 - \varepsilon_4, 1 + \varepsilon_4) \quad x \in \mathbb{R} \setminus \frac{1}{2}(ln_{51}, rn_{51}), \text{ in particular for } z = ln_{51}, rn_{51}; \quad (4.3)$$

$$r_{s_{52}}(z) \in (1 - \varepsilon_4, 1 + \varepsilon_4) \quad x \in \mathbb{R} \setminus \frac{1}{2}(ln_{52}, rn_{52}), \text{ in particular for } z = ln_{52}, rn_{52}. \quad (4.4)$$

Put $E_{11} := \cup_{i=1}^9 s_i \cup s_{51} \cup s_{52}$.

Now we have 10 complementary intervals to E_{11} on $[-1, 1]$. Assign p_5, p_6 to left and right endpoints of segment s_5 . This will make measure μ_{11} very small on s_5 . Recall that μ_5 was quite large on s_5 —so what we are doing is the process of removal of mass from s_5 . Where this mass will disappear? To understand this first let us assign the points p_7, \dots, p_{10} .

We have a complementary interval between segments s_6 and s_{51} . Put p_7 in its left endpoint (=right endpoint of s_6).

We have a complementary interval between segments s_{51} and s_7 . Put p_8 in its right endpoint (=left endpoint of s_7).

We have a complementary interval between segments s_8 and s_{52} . Put p_9 in its left endpoint (=right endpoint of s_8).

We have a complementary interval between segments s_{52} and s_9 . Put p_{10} in its right endpoint (=left endpoint of s_9).

This assignment fixes function $R_{11}(z) = \int_{E_{11}} \frac{d\mu_{11}(x)}{x-z}$.

As a result of the construction mass almost disappears from s_5 :

$$\mu_{11}(s_5) \leq (1 + \varepsilon_1)(1 + \varepsilon_2)^2(1 + \varepsilon_3)^4 \int_{-\ell_1}^{\ell_1} \sqrt{\ell_1^2 - x^2} dx \asymp \ell_1^3. \quad (4.5)$$

Let us see that it reappears on s_{51}, s_{52} in almost equal parts. Let us denote temporarily the endpoints of s_{51} by a, b , endpoints of s_5 by c, d and endpoints s_{52} by e, f . We do not need endpoints of $s_k, k = 1, 2, \dots, 9$ as $r_{s_k}(z) \in (1 - \varepsilon_i, 1 + \varepsilon_i)$ for $i = 1, 2, 3$ for $z \in s_{51}$ by construction. Then for $x \in s_{51}$ we clearly have

$$\frac{d\mu_{11}(x)}{dx} / \left[\frac{\sqrt{(c-x)(d-x)}}{\sqrt{(e-x)(f-x)}} \frac{1}{\sqrt{(b-x)(x-a)}} \right] \in ((1-\varepsilon_1)(1-\varepsilon_2)^2(1-\varepsilon_3)^2, (1+\varepsilon_1)(1+\varepsilon_2)^2(1+\varepsilon_3)^2) \quad (4.6)$$

But from (4.2) and (4.4) we get

$$\frac{\sqrt{(c-x)(d-x)}}{\sqrt{(e-x)(f-x)}} \in \frac{1}{2}((1-\varepsilon_3)^2(1-\varepsilon_4)^2, (1+\varepsilon_3)^2(1+\varepsilon_4)^2). \quad (4.7)$$

Combining (4.6), (4.7) we get

$$\mu_{11}(s_5 1) \geq \frac{1}{2} \prod_{i=1}^4 (1 - \varepsilon_i)^{2^i} \int_{-\ell_2}^{\ell_2} \frac{dx}{\sqrt{\ell_2^2 - x^2}} \asymp \frac{1}{2} \mu_5(s_5). \quad (4.8)$$

Symmetrically

$$\mu_{11}(s_5 2) \geq \frac{1}{2} \prod_{i=1}^4 (1 - \varepsilon_i)^{2^i} \int_{-\ell_2}^{\ell_2} \frac{dx}{\sqrt{\ell_2^2 - x^2}} \asymp \frac{1}{2} \mu_5(s_5). \quad (4.9)$$

Notice that

- Green's function G_{11} of $\Omega_{11} := \mathbb{C} \setminus E_{11}$ is smaller than ε_3 between s_6, s_7 and between s_8, s_9 .
- Measure μ_{11} built by our formula on E_{11} by assignment of ten points p_1, \dots, p_{10} has very large mass on s_{51}, s_{52} , in fact these masses are $(\frac{1}{2} - \frac{1}{8})\mu_5(s_5)$.

Renaming. We use name n_{51} for n_3 , name n_{52} for n_4 and even n_5 for n_2 . We call s_6 — Ln_{51} , s_7 — Rn_{51} , s_8 — Ln_{52} , s_9 — Rn_{52} . We rename E_5 into E^1 , it has 5 segments, $\Omega^1 := \mathbb{C} \setminus E^1$ (it was called Ω_5 before). Also E^2 is now is E_{11} it has $5 + 2 + 2^2$ segments, $\Omega_2 = \mathbb{C} \setminus E^2$.

Next steps. We are going to repeat previous steps. We put a very small n_{511} centered at ln_{51} , n_{512} centered at rn_{51} , n_{521} centered at ln_{52} , n_{522} centered at rn_{52} . (These $ln_{5i}, rn_{5j}, i, j = 1, 2$ are midpoints of left and right halves of n_{5i} . We build $Ln_{5ij}, Rn_{5ij}, i, j = 1, 2$ as before. And then we build $s_{5ij}, i, j = 1, 2$. We get E^3 with $5 + 2 + 2^2 + 2^2 + 2^3$ segments, then E^4 with

$5 + 2 + 2^2 + 2^2 + 2^3 + 2^3 + 2^4$ segments. Measures are called μ^k now, so μ^1 is μ_5 and μ^2 is μ_{11} . Et cetera.... We already saw that

$$\mu_2(s_{5i}) \geq \left(\frac{1}{2} - \frac{1}{8}\right)\mu^1(s_5), \quad \forall i = 1, 2. \quad (4.10)$$

By construction will also have

$$\mu_3(s_{5ij}) \geq \left(\frac{1}{2} - \frac{1}{8}\right)\left(\frac{1}{2} - \frac{1}{16}\right)\mu^1(s_5), \quad \forall i, j = 1, 2. \quad (4.11)$$

Similarly,

$$\mu_4(s_{5ijk}) \geq \left(\frac{1}{2} - \frac{1}{8}\right)\left(\frac{1}{2} - \frac{1}{16}\right)\left(\frac{1}{2} - \frac{1}{32}\right)\mu^1(s_5), \quad \forall i, j = 1, 2. \quad (4.12)$$

Et cetera...

So on the union of n -th generation $s_{5i_1 \dots i_n}$ we still have measure μ^{n+1} having a mass bigger than numerical positive constant c_0 .

On the other hand $\mu^{n+2}, \mu^{n+3}, \dots$ are small on the union of these intervals.

Therefore the limit measure μ has mass c_0 **not** on the union of constructed countably many intervals of all generations. So this measure is of the **limit set** of these intervals, so this measure has a singular component of mass at least c_0 .

Why this singular component does not have point component?

This is because we always divide the measure of each segment $s_{5i_1 \dots i_n}$ into two almost equal parts. Hence one can easily see that

$$\lim_{|I| \rightarrow 0} \mu(I) = 0. \quad (4.13)$$

Of course we need

$$\sum 2^i \varepsilon_i < \infty.$$

Remark 1. We can also construct Widom domain $\Omega = \mathbb{C} \setminus E$ and distribute p_j in such a way that measure ν in the representation

$$-1/R(z) - z - c = \int_E \frac{d\nu(x)}{x - z}$$

has a continuous singular part, and it is still such that $R(x + i0)$ is purely imaginary dx a.e. on E .

Next is a curious remark about **homogeneous** sets E ! Recall that for homogeneous sets E reflectionless measures μ always give rise to H^1 functions R and all reflectionless measures supported on homogeneous set E and having it as its abs. continuous spectrum cannot have any singular component! However,

Remark 2. We can also construct domain $\Omega = \mathbb{C} \setminus E$ with **homogeneous** E and distribute $p_j < q_j$ in such a way that after constructing R by p_j and Q by q_j we get R/Q to be a Nevanlinna function and the measure ν in the representation

$$\frac{R(z)}{Q(z)} = \int_E \frac{d\nu(x)}{x - z}$$

has a continuous singular part.

REFERENCES

- [MPV] M. MELNIKOV, A. PLTORATSKII, A. VOLBERG, *Uniqueness theorems for Cauchy integral*, arXiv, 2007.
- [SoYu] M. SODIN, P. YUDITSKII, *Almost periodic Jacobi matrices with homogeneous spectrum, infinite dimensional Jacobi inversion, and Hardy spaces of character automorphic functions*. J. of Geom. Analysis, **7** (1997), 387–435.
- [Z] M. ZINSMEISTER, *Espaces de Hardy et domaines de Denjoy*, Ark. Mat. **27** (1989), 363–378.